Distance Distributions and Boundary Effects in Finite Uniformly Random Networks

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Jan. 2013

Outline

- **Motivation and Background**
  - Spatial Point Processes
  - Distance Distributions
- **Prior Work**
  - Poisson Point Process (PPP)
  - Binomial Point Process (BPP)
- **Problem Formulation**
  - Modelling of Boundary Effects
  - Proposed Algorithm
- **Results**
- **Conclusions**

Background

- **Spatial point processes** are used to model the locations of objects or events in a wide variety of scientific disciplines*.
  - **Forestry/Seismology/Geography/Astronomy**
    - Locations of trees/earthquake epicenters/cities/galaxies
  - **Medicine and Biology**
    - Home locations of infected patients.
    - Spikes of neurons.
    - Microcalcifications in mammogram images.
  - **Material Science**
    - Positions of defects in industrial materials.

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*A. Baddeley, "Analysing spatial point patterns in R", *CSIRO Workshop Notes*, Feb. 2008. [**Cited by 71**]

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- **Material Science**
  - Positions of defects in industrial materials.
- **Wireless Communications**
  - A wireless network can be viewed as a collection of nodes, where the **location of nodes** are seen as realizations of some spatial point process.

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Background - PPP

- Popular model: **infinite homogenous Poisson point process (PPP)**.
  - **Rationale**: Homogeneous PPP can be regarded as the limiting case of a uniform distribution of N nodes on an area of size A, as N and A tend to infinity but their ratio $\rho = N/A$ remains constant.

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  - **Rationale**: Homogeneous PPP can be regarded as the limiting case of a uniform distribution of N nodes on an area of size A, as N and A tend to infinity but their ratio $\rho = N/A$ remains constant.
  - **Advantage**: Mathematical tractability - provides a model for 'completely random' distribution of points.
  - **Main shortcoming**: The number of nodes in disjoint areas is independent.

Background - BPP

- More realistic model: Finite number of nodes independently and uniformly distributed over a finite area (**Binomial point process (BPP)**).
  - Cellular networks: cells are hexagons.
  - Ad hoc and sensor networks: finite square region.

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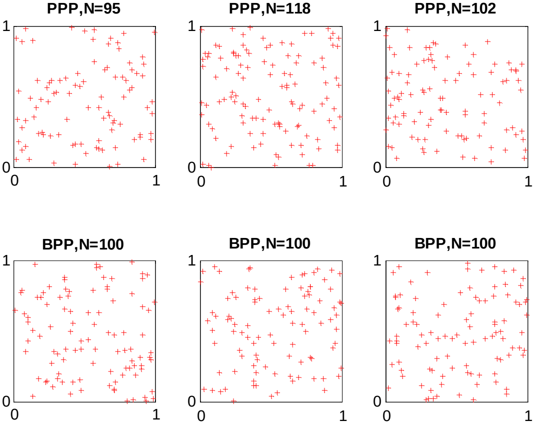
  - Cellular networks: cells are hexagons.
  - Ad hoc and sensor networks: finite square region.

- **Advantage**: The number of nodes in disjoint areas is no longer independent: the more nodes in one sub-area, the fewer can fall in another.

Background - PPP & BPP

◇ Illustration: Nodes distributed in 1×1 m^2 area according to (a) PPP, 100 nodes/m^{2†} and (b) BPP, $N = 100$.

PPP in 2D can be realized as a 1D PPP enriched by attaching to each one-dimensional point an independent Uniform random variable to provide the second coordinate.



BPP in 2D:
»x=rand(1,100);
»y=rand(1,100);
»plot(x,y,'r+');

† Sheldon M. Ross, *Simulation*, 4th ed., Elsevier Inc., 2006.

Background - Spatial Point Processes

◇ Useful point processes for wireless network modeling‡:

Point Process	Key Properties	Practical Example
Poisson (PPP)	Mutual independence between (transmitting) node locations.	Ad hoc networks with pure random channel access.
Binomial	Similar to PPP as far as i.i.d. node locations, but with a fixed number of nodes in a given area.	A known number of relays or mobile users deployed at random in a cell of known size
Poisson cluster (PCP)	Clustering of nodes, with independence between cluster locations.	Sensor networks, military platoons, an urban network with dense hotspots.
Poisson plus Poisson Cluster	Independence between the PCP and the PPP. Attraction between nodes.	PPP represents the mobile users in a macrocell and the PCP represents femtocells or hotspots.
Matern hard core	Minimum distance between nodes.	Carrier sensing wireless networks with collision avoidance, e.g. WiFi.
Determinantal	Repulsion between nodes, e.g. Ginibre Process.	CSMA networks, networks with soft minimum distance.

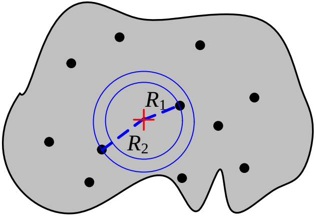
‡ J. G. Andrews et. al., "A primer on spatial modeling and analysis in wireless networks", *IEEE Communications Magazine*, vol. 48, no. 9, pp. 156—163, Nov. 2010. [**Cited by 42**]

Distance Distributions

◇ The performance of wireless networks depends critically on the **distances** between the transmitters and receivers.

Distance Distributions

- The performance of wireless networks depends critically on the **distances** between the transmitters and receivers.
- Euclidean distance to ***n*-th neighbor** from an arbitrarily chosen reference point.
  - $n = 1$ corresponds to nearest neighbour.
  - $n = 2$ corresponds to second nearest neighbour.
  - $n = N$ corresponds to farthest neighbour.



n-th Neighbour PDF – PPP

- PDF of Euclidean distance to n-th nearest neighbor in a homogeneous m-dimensional PPP: **generalized Gamma distribution**§

$$f_{R_n}(r) = \frac{m(\rho c_m r^m)^n}{r\Gamma(n)} e^{-\rho c_m r^m}$$

where coefficients c_m are given by

$$c_m = \begin{cases} \dfrac{\pi^{\frac{m}{2}}}{\left(\frac{m}{2}\right)!} & \text{for even } m \\ \dfrac{\pi^{\frac{m-1}{2}}\left(\frac{m-1}{2}\right)!}{m!} & \text{for odd } m \end{cases}$$

(e.g., $c_1 = 2, c_2 = \pi,\ c_3 = \frac{4\pi}{3}$)

§ M. Haenggi, "On Distances in Uniformly Random Networks", *IEEE Trans. Inf. Theory*, vol. 51, no. 10, pp. 3584–3586, 2005. [**Cited by 166**]

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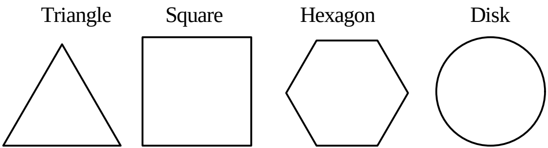
(e.g., $c_1 = 2, c_2 = \pi,\ c_3 = \frac{4\pi}{3}$)

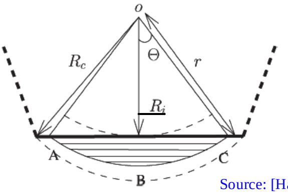
◇ **Special case ($m = 2,\ n = 1$)**: $f_{R_1}(r) = 2\pi\rho r e^{-\rho\pi r^2}$

§ M. Haenggi, "On Distances in Uniformly Random Networks", *IEEE Trans. Inf. Theory*, vol. 51, no. 10, pp. 3584–3586, 2005. **[Cited by 166]**

Distance Distributions – BPP

◇ **Distance distribution in a BPP with N nodes distributed inside a L-sided regular polygon (L-gon) with area $\mathcal{A}$.**





Source: [Haenggi Paper]

Section of an l-sided regular polygon depicting one of its sides.

Distance Distributions – BPP

- **Distance distribution for BPP in a polygon (assuming center of polygon as reference point)**[¶]

$$f_{R_n}(r) = \begin{cases} \frac{2r\pi}{A} \frac{(1-p)^{N-n} p^{n-1}}{B(N-n+1,n)} & 0 < r \leq R_i \\ \frac{2r(\pi - L\theta)}{A} \frac{(1-q)^{N-n} q^{N-n}}{B(N-n+1,n)} & R_i < r \leq R_c \\ 0 & R_c < r \end{cases}$$

where

$R_i = \sqrt{\frac{A}{L} \cot\left(\frac{\pi}{L}\right)}$,

$R_c = \sqrt{\frac{2A}{L} \csc\left(\frac{2\pi}{L}\right)}$,

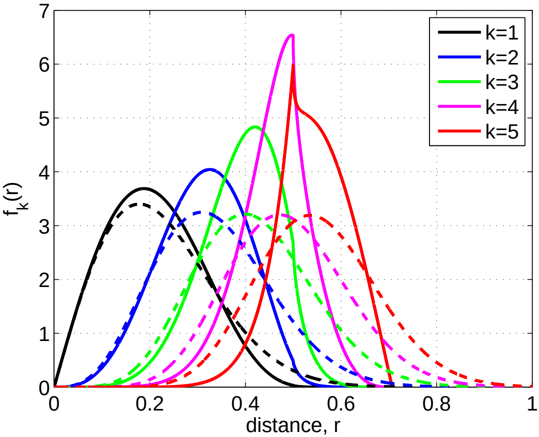
$p = \frac{\pi r^2}{A}$, $q = \frac{\pi r^2 - \left(L r^2 \theta - L R_i \sqrt{r^2 - R_i^2}\right)}{A}$, $\theta = \arccos(R_i / r)$,

beta function $B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$.

[¶] S. Srinivasa and M. Haenggi, "Distance Distributions in Finite Uniformly Random Networks: Theory and Applications", *IEEE Trans. Veh. Tech.*, vol. 59, no. 2, pp. 940–949, Feb. 2010. **[Cited by 34]**

Distance Distributions - Illustration

- BPP with $N = 5$ nodes distributed inside a unit square ($L = 4$) (solid lines = BPP, dotted lines = PPP).



Contribution of this Work

- **We derive the closed-form PDF of the distance between any arbitrary reference point and its n-th neighbour node, when N nodes are uniformly distributed inside a regular L-sided polygon.**

Polygon Geometry

- N nodes are independently and uniformly distributed inside a regular L-sided polygon $\mathcal{A}$, inscribed in a circle of radius R centered at the origin. Let $\boldsymbol{u} = [x, y]^T$ denote an arbitrary reference point.

Circumradius: R

Inradius: $R_{\mathrm{i}} = R\cos(\pi/L)$

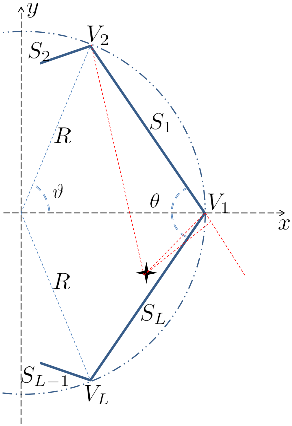
Area:

$A = |\mathcal{A}| = \frac{1}{2}LR^2 \sin\left(\frac{2\pi}{L}\right)$

Side length: $t = 2R\sin\left(\frac{\pi}{L}\right)$

Interior angle: $\theta = \frac{\pi(L-2)}{L}$

Central angle: $\vartheta = \frac{2\pi}{L}$

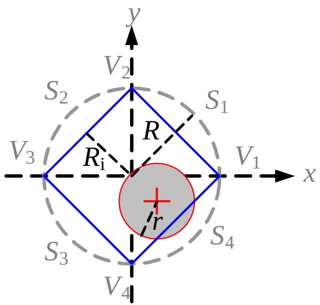


Problem Formulation

- Define the **cumulative density function (CDF)** $F(\boldsymbol{u}; r)$, which is the probability that a random node falls inside a disk $\mathcal{D}(\boldsymbol{u}; r)$ centered at the arbitrary reference point $\boldsymbol{u}$, as

$$F(\boldsymbol{u}; r) = \frac{|\mathcal{D}(\boldsymbol{u}; r) \cap \mathcal{A}|}{|\mathcal{A}|} = \frac{O(\boldsymbol{u}; r)}{A}$$

where $O(\boldsymbol{u}; r) = |\mathcal{D}(\boldsymbol{u}; r) \cap \mathcal{A}|$ is the overlap area.



Problem Formulation

- The **CCDF** expressing the probability that there are less than n nodes in the disk $\mathcal{D}$ is given by[‖]

$$1 - F_n(\boldsymbol{u}; r) = \sum_{j=0}^{n-1} \binom{N}{j} (F(\boldsymbol{u}; r))^j (1 - F(\boldsymbol{u}; r))^{N-j}$$

‖ S. Srinivasa and M. Haenggi, "Distance Distributions in Finite Uniformly Random Networks: Theory and Applications", *IEEE Trans. Veh. Tech.*, vol. 59, no. 2, pp. 940–949, Feb. 2010. [**Cited by 34**]

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- The corresponding **PDF** is

$$f_n(r) = \frac{(1 - F(\boldsymbol{u}; r))^{N-n} (F(\boldsymbol{u}; r))^{n-1}}{B(N-n+1, n)} \frac{d}{dr} F(\boldsymbol{u}; r)$$

where beta function $B(a, b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$.

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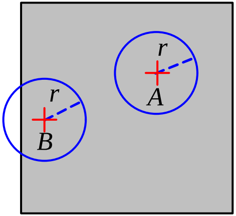
Why is the CDF $F(\boldsymbol{u}; r)$ so hard to find ?

◇ **Boundary effects**: nodes located near the physical boundaries of the region have their coverage area reduced.**

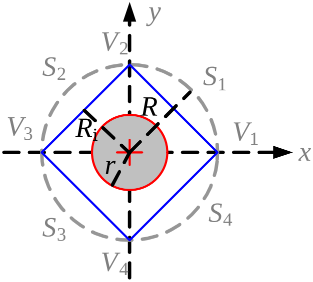


** C. Bettstetter, "On the minimum node degree and connectivity of a wireless multihop network", in *Proc. 3rd ACM international symposium on Mobile ad hoc networking & computing*, 2002. **[Cited by 727]**

Special case: reference point at the center

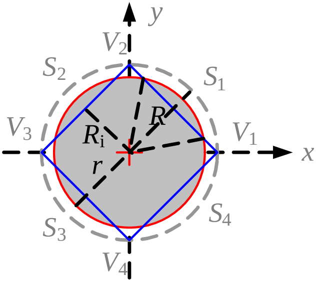
◇ **Boundary effects are easy to characterise**: circular segment areas are symmetric, with no overlap.

$0 \leq r \leq R_\mathrm{i}$



$$F(\mathbf{u}; r) = \frac{\pi r^2}{A}$$

$R_\mathrm{i} \leq r \leq R$

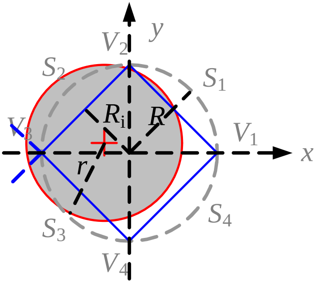


$$F(\mathbf{u}; r) = \frac{\pi r^2 - 4(\text{area of one circular segment})}{A}$$

General case: reference point at arbitrary location

◇ **Boundary effects are complicated to characterise**:

- **Problem 1**: circular segment areas are no longer symmetric and they may have overlap.



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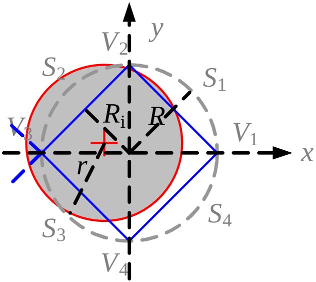
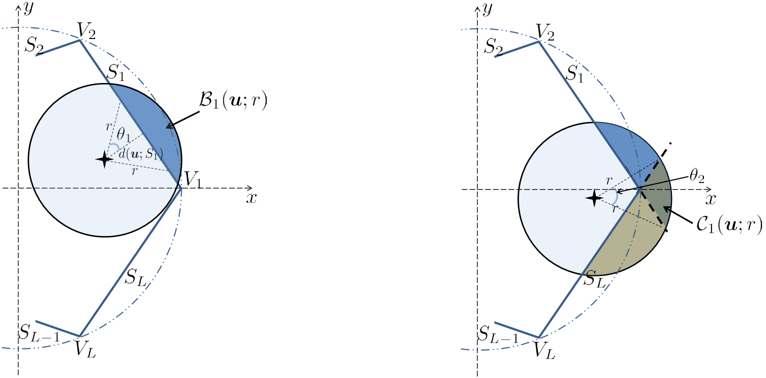


  - **Problem 2**: since a L-gon has L sides and L vertices, there can be $2 * L + 1$ different ranges for the distance r.

Proposed Approach

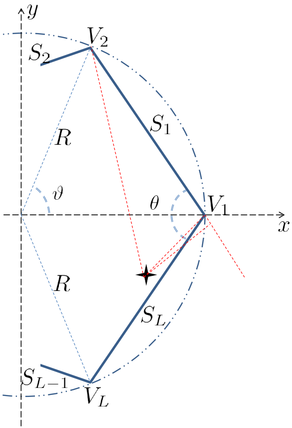
- We decompose the **boundary effects into border and corner effects**.
  - Let $B_\ell =$ the area of the circular segment formed outside the side S_ℓ $(\ell = 1, 2, \ldots, L)$.
  - Let $C_\ell =$ the corner overlap area formed at vertex V_ℓ.



Distances to Sides and Vertices

- Distance $d(\boldsymbol{u};\, V_1)$ between the point $\boldsymbol{u}$ and the vertex V_1 is

$$d(\boldsymbol{u};\, V_1) = \sqrt{(x - R)^2 + y^2}$$



Distances to Sides and Vertices

◇ Shortest distance $d(\boldsymbol{u}; S_1)$ to the side S_1 is

$$d(\boldsymbol{u}; S_1) = \begin{cases} \min(d(\boldsymbol{u}; V_1), d(\boldsymbol{u}; V_2)), & \max(d(\boldsymbol{w}; V_1), d(\boldsymbol{w}; V_2)) > t; \\ p(\boldsymbol{u}; S_1), & \text{otherwise;} \end{cases}$$

where t is the side length and

$$\boldsymbol{w} = \left[R - \frac{1}{2}(x - R)(\cos\vartheta - 1) + y\sin\vartheta, \frac{\sin\vartheta\big((x - R)(\cos\vartheta - 1) + y\sin\vartheta\big)}{2(1 - \cos\vartheta)}\right]$$

$$p(\boldsymbol{u}; S_1) = \frac{\text{abs}\left(y + \tan\left(\frac{\theta}{2}\right)x - R\tan\left(\frac{\theta}{2}\right)\right)}{\sqrt{1 + \tan^2\left(\frac{\theta}{2}\right)}}$$

Corner Effects

- **Circular segment area** formed outside side S_1 of ℓ-gon is[††]

$$B_1(\boldsymbol{u};r) = \begin{cases} r^2 \arccos\left(\frac{p(\boldsymbol{u};S_1)}{r}\right) - (d(\boldsymbol{u};S_1))^2 \arccos\left(\frac{p(\boldsymbol{u};S_1)}{d(\boldsymbol{u};S_1)}\right) - \\ \quad p(\boldsymbol{u};S_1)\left(\sqrt{r^2 - \left(p(\boldsymbol{u};S_1)\right)^2} - \sqrt{(d(\boldsymbol{u};S_1))^2 - \left(p(\boldsymbol{u};S_1)\right)^2}\right), & r \geq d(\boldsymbol{u};S_1); \\ 0, & \text{otherwise}; \end{cases}$$

- **Corner overlap area** formed at vertex V_1 of ℓ-gon is

$$C_1(\boldsymbol{u};r) = \begin{cases} \frac{r^2}{2}\left(\arccos\left(\frac{p(\boldsymbol{u};S_1)}{r}\right) + \arccos\left(\frac{p(\boldsymbol{u};S_L)}{r}\right)\right) - \\ \frac{(d(\boldsymbol{u};V_1))^2}{2}\left(\arccos\left(\frac{p(\boldsymbol{u};S_1)}{d(\boldsymbol{u};V_1)}\right) + \arccos\left(\frac{p(\boldsymbol{u};S_L)}{d(\boldsymbol{u};V_1)}\right)\right) + \\ \frac{p(\boldsymbol{u};S_1)}{2}\left(\sqrt{\left(d(\boldsymbol{u};V_1)\right)^2 - \left(p(\boldsymbol{u};S_1)\right)^2} - \sqrt{r^2 - \left(p(\boldsymbol{u};S_1)\right)^2}\right) + \\ \frac{p(\boldsymbol{u};S_L)}{2}\left(\sqrt{\left(d(\boldsymbol{u};V_1)\right)^2 - \left(p(\boldsymbol{u};S_L)\right)^2} - \sqrt{r^2 - \left(p(\boldsymbol{u};S_L)\right)^2}\right) - \\ \frac{\pi}{L}\left(r^2 - \left(d(\boldsymbol{u};V_1)\right)^2\right), & r \geq d(\boldsymbol{u};V_1); \\ 0, & \text{otherwise}; \end{cases}$$

[††] Z. Khalid and S. Durrani, "Distance Distributions in Regular Polygons", *IEEE Trans. Veh. Tech.*, 2013 (in press: http://arxiv.org/abs/1207.5857).

Rotation Operator

- We define the rotation operator $\mathfrak{R}^{\ell}$ which rotates an arbitrary point $\boldsymbol{u} = [x, y]^T$ anti-clockwise around the origin by an angle $\ell\vartheta$.

- The rotated point $\mathfrak{R}^{\ell}\boldsymbol{u}$ can be expressed as

$$(\mathfrak{R}^{\ell}\boldsymbol{u}) = \mathbf{T}\,\boldsymbol{u}$$

- The rotation matrix is given by

$$\mathbf{T} = \begin{pmatrix} \cos(\ell\vartheta) & -\sin(\ell\vartheta) \\ \sin(\ell\vartheta) & \cos(\ell\vartheta) \end{pmatrix}$$

Exploiting Rotational Symmetry

◇ Solution to **Problem 1: circular segment areas are no longer symmetric and they may have overlap**:

$$d(\boldsymbol{u};\, V_\ell) = d(\mathfrak{R}^{-(\ell-1)}\boldsymbol{u};\, V_1)$$

$$p(\boldsymbol{u};\, S_\ell) = p(\mathfrak{R}^{-(\ell-1)}\boldsymbol{u};\, S_1)$$

$$d(\boldsymbol{u};\, S_\ell) = d(\mathfrak{R}^{-(\ell-1)}\boldsymbol{u};\, S_1)$$

$$B_\ell(\boldsymbol{u}; r) = \begin{cases} B_1(\mathfrak{R}^{-(\ell-1)}\boldsymbol{u}; r), & r \geq d(\boldsymbol{u}; S_\ell); \\ 0, & \text{otherwise.} \end{cases}$$

$$C_\ell(\boldsymbol{u}; r) = \begin{cases} C_1(\mathfrak{R}^{-(\ell-1)}\boldsymbol{u}; r), & r \geq d(\boldsymbol{u}; V_\ell); \\ 0, & \text{otherwise.} \end{cases}$$

Exploiting Rotational Symmetry

- Define the distance vector **d** as

$$\mathbf{d} = [d(\boldsymbol{u}; S_1), \ldots, d(\boldsymbol{u}; S_L), d(\boldsymbol{u}; V_1), \ldots, d(\boldsymbol{u}; V_L)]$$

and $\acute{\mathbf{d}}$ is the sorted distance vector in ascending order. **k** is the index vector that transforms **d** into $\acute{\mathbf{d}}$.

- We use the sorted distance vector $\acute{\mathbf{d}}$ and the index vector **k** to identify each unique range and to find the boundary effects for that range.

Proposed Algorithm

◇ Solution to **Problem 2: there can be $2*L+1$ different ranges for the distance r**:

Algorithm 1 Algorithm to find the overlap area

Step 1: Sort d in (10) in ascending order to obtain d́
Step 2: Determine the index sorting that transforms d into d́ and obtain the index vector k
Step 3: Find the appropriate circular segment areas and the overlap area

```
for each j in j = 1, 2, 3, ..., 2L + 1 do
  if d́_{j-1} − d́_j ≠ 0, (d́_0 = 0) then
    O_j(u; r) = πr^2
    for each i in i = 1, 2, 3, ..., j − 1 do
      if k_i ≤ L then
        O_j(u; r) = O_j(u; r) − B_{k_i}(u; r)
      else
        O_j(u; r) = O_j(u; r) + C_{k_i−L}(u; r)
      end if
    end for
  end if
end for
```

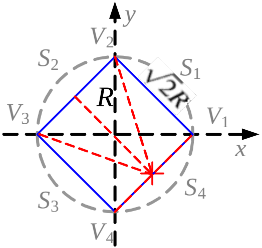
Results: Example 1

◇ **Arbitrary reference point: middle of side S_4 for a square with $R = 1$.**

$$\mathbf{d} = \left[\frac{R}{\sqrt{2}}, \sqrt{2}R, \frac{R}{\sqrt{2}}, 0, \frac{R}{\sqrt{2}}, \frac{\sqrt{10}R}{2}, \frac{\sqrt{10}R}{2}, \frac{R}{\sqrt{2}}\right]$$

$$\acute{\mathbf{d}} = \left[0, \frac{R}{\sqrt{2}}, \frac{R}{\sqrt{2}}, \frac{R}{\sqrt{2}}, \frac{R}{\sqrt{2}}, \sqrt{2}R, \frac{\sqrt{10}R}{2}, \frac{\sqrt{10}R}{2}\right]$$

$$\mathbf{k} = [4, 1, 3, 5, 8, 2, 6, 7]$$



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$$\acute{\mathbf{d}} = \left[0, \frac{R}{\sqrt{2}}, \frac{R}{\sqrt{2}}, \frac{R}{\sqrt{2}}, \frac{R}{\sqrt{2}}, \sqrt{2}R, \frac{\sqrt{10}R}{2}, \frac{\sqrt{10}R}{2}\right]$$

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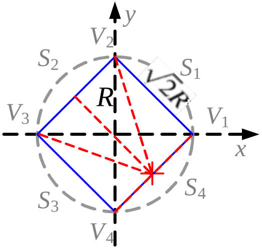
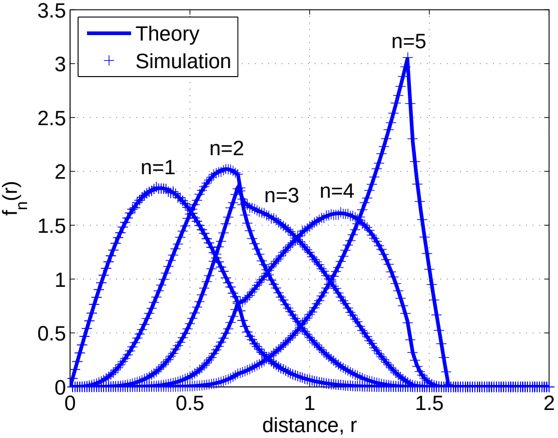


◇ The CDF is

Range	$F(\boldsymbol{u}; r)$
$0 \leq r \leq R/\sqrt{2}$	$\frac{\pi r^2 - (B_4)}{A}$
$R/\sqrt{2} \leq r\sqrt{2}R$	$\frac{\pi r^2 - (B_1 + B_3 + B_4 - C_1 - C_4)}{A}$
$\sqrt{2}R \leq r \leq \sqrt{10}R/2$	$\frac{\pi r^2 - (B_1 + B_2 + B_3 + B_4 - C_1 - C_2)}{A}$
$r \geq \sqrt{10}R/2$	1

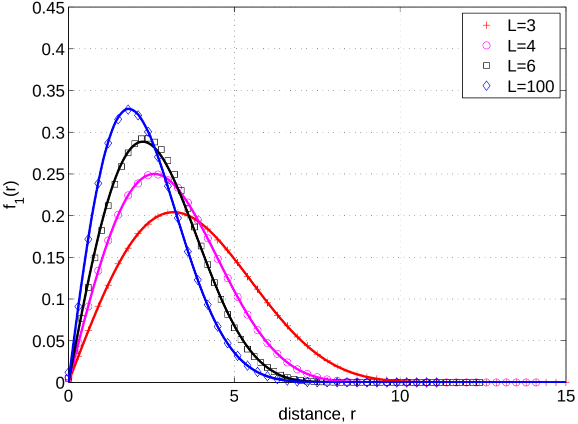
Results: Example 1

◇ **Arbitrary reference point: middle of side S_4 for a square with $R = 1$ and $N = 5$.**



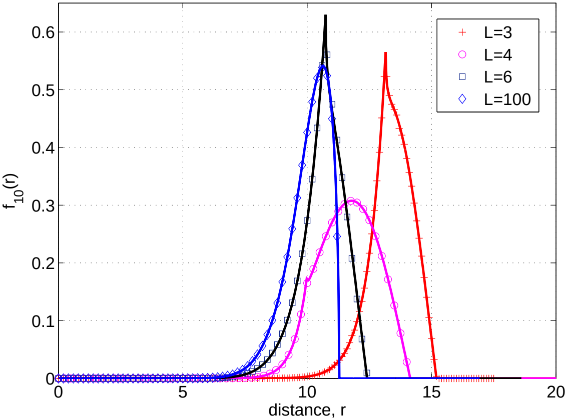
Results: Example 2

- **Arbitrary reference point located at vertex of a polygon with Area $A = 100$ and $N = 10$ nodes: PDF of nearest neighbour.**



Results: Example 3

◇ **Arbitrary reference point located at vertex of a polygon with Area $A = 100$ and $N = 10$ nodes: PDF of farthest neighbour**.



Conclusion and Future Work

- In this work, we have derived the *n*-th neighbour distance distribution results in regular polygons.

- The knowledge of these **general distance distributions** can be used to analyse the wireless network characteristics from the perspective of an arbitrary node located **anywhere** (i.e. not just the center) in the finite coverage area.

- **Applications**:
  - Connectivity: S. Durrani, Z. Khalid and J. Guo, "A Tractable Framework for Exact Probability of Node Isolation in Finite Wireless Sensor Networks", *submitted to IEEE Trans. Veh. Tech.*, 2013 (http://arxiv.org/abs/1212.1283)
  - Interference and outage: work under progress.

Thank you for your attention